A model for interacting instabilities and texture dynamics of patterns

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A simple model to study interacting instabilities and textures of resulting patterns for thermal convection is presented. The model consisting of twelve-mode dynamical system derived for periodic square lattice describes convective patterns in the form of stripes and patchwork quilt. The interaction between stationary zig-zag stripes and standing patchwork quilt pattern leads to spatiotemporal patterns of *twisted* patchwork quilt. Textures of these patterns, which depend strongly on Prandtl number, are investigated numerically using the model. The model also shows an interesting possibility of a multicritical point, where stability boundaries of four different structures meet.

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Pattern-forming instabilities in systems driven externally far from equilibrium are currently receiving considerable attention [1-11]. They appear in many physical systems such as fluids [3-8], granular materials [9], cardiac tissues [10], reaction-diffusion systems [11], traffic flow [12], dendritic growth [13], and non-linear optics [14]. Spatiotemporal structures arising due to interacting instabilities and the dynamics of their textures are understood theoretically either by amplitude equations [1-2] or dynamical systems [15]. In absence of a clear separation of time scales, dynamical systems are preferred for investigating pattern dynamics. Roberts *et al* [16], using a dynamical system with hexagonal symmetry, showed the possibility of standing patchwork quilt pattern due to interacting oscillatory instabilities in the problem of thermal convection in a *double-diffusive* system [17]. They found mirror-symmetric and *twisted* patchwork quilt patterns on hexagonal lattice. Patterns having both open and closed streamlines are called patchwork quilt. *Twisted* patchwork quilt does not have mirror symmetry.

In this work, we present a simple model of interacting instabilities in the form of a twelve-mode dynamical system derived from Boussinesq equations for thermal convection in *ordinary* fluids. Using the model, we show that the interaction between zig-zag pattern and standing squares can also lead to *twisted* patchwork quilt pattern. Our model is based on square lattice rather than hexagonal lattice, and it requires only two bifurcation parameters: the Prandtl number σ and reduced Rayleigh number r. The possibility of patchwork quilt patterns on square lattice due to interaction of a stationary and an oscillatory instabilities is qualitatively new. We then investigate numerically textures of spatiotemporal structures arising due to interacting patterns. The model also shows an interesting possibility of a multicritical point ($\sigma = 1.57 \pm 0.01,\ r = 11.2 \pm 0.05$), where stability zones of straight stripes, zig-zag stripes, standing *symmetric* patchwork quilt, and standing *twisted* patchwork quilt meet.

We consider an extended horizontal layer of Boussinesq fluid of thickness d, kinematic viscosity ν, thermal diffusivity κ confined between two perfectly conducting *stress-free* horizontal boundaries, and heated from below. Making all length scales dimensionless by the fluid thickness d, time by the thermal diffusive time scale d^2/κ, and the temperature by the temperature difference ΔT between the two bounding surfaces, the relevant hydrodynamical equations in dimensionless form read

$$\begin{aligned}
\partial_t \nabla^2 v_3 &= \sigma\nabla^4 v_3 + \sigma\nabla_H^2\theta \\
&- \mathbf{e_3}.[\nabla\times\{(\omega.\nabla)\mathbf{v} - (\mathbf{v}.\nabla)\omega\}], \qquad (1)\\
\partial_t \omega_3 &= \sigma\nabla^2\omega_3 + [(\omega.\nabla)v_3 - (\mathbf{v}.\nabla)\omega_3], \qquad (2)\\
\partial_t\theta &= \nabla^2\theta + Rv_3 - \mathbf{v}.\nabla\theta, \qquad (3)
\end{aligned}$$

where $\mathbf{v} \equiv (v_1, v_2, v_3)$, $\omega = \nabla\times\mathbf{v} \equiv (\omega_1, \omega_2, \omega_3)$, and θ are respectively the velocity, the vorticity, and the deviation from the conductive temperature profile. Prandtl number σ and Rayleigh number R are defined respectively as $\sigma = \frac{\nu}{\kappa}$ and $R = \frac{\alpha(\Delta T)gd^3}{\nu\kappa}$, where α is the coefficient of thermal expansion of the fluid, g the acceleration due to gravity. The unit vector $\mathbf{e_3}$ is directed vertically upward. The symbol $\nabla_H^2 (= \nabla_{11}+\nabla_{22})$ stands for horizontal Laplacian. The boundary conditions at the idealized *stress-free* conducting flat surfaces imply $\theta = v_3 = \partial_{33}v_3 = \partial_3\omega_3 = 0$ at $x_3 = 0, 1$.

We construct a dynamical system by standard Galerkin procedure. The spatial dependence of vertical velocity, vertical vorticity and temperature field

are expanded in a Fourier series, which is compatible with the stress-free flat conducting boundaries and periodic square lattice in the horizontal plane. We include minimum modes to describe straight stripes (S), zig-zag stripes (ZZ), square patterns (SQ), and nonlinear interaction among these patterns. The vertical velocity v_3, vertical vorticity w_3, and θ then may be written as

$$\begin{aligned} v_3 &= [W_{101}(t)\cos k_c x_1 + W_{011}(t)\cos k_c x_2]\sin \pi x_3 \\ &+ [W_{112}(t)\cos k_c x_1 \cos k_c x_2 \\ &\quad + W_{\bar{1}\bar{1}2}\sin k_c x_1 \sin k_c x_2]\sin 2\pi x_3, \qquad (4) \\ \omega_3 &= [\zeta_{101}(t)\cos k_c x_1 + \zeta_{011}(t)\cos k_c x_2]\cos \pi x_3 \\ &+ \zeta_{110}(t)\cos k_c x_1 \cos k_c x_2, \qquad (5) \\ \theta &= [\Theta_{101}(t)\cos k_c x_1 + \Theta_{011}(t)\cos k_c x_2]\sin \pi x_3 \\ &+ \Theta_{002}(t)\sin 2\pi x_3 + [\Theta_{112}(t)\cos k_c x_1 \cos k_c x_2 \\ &\quad + \Theta_{\bar{1}\bar{1}2}\sin k_c x_1 \sin k_c x_2]\sin 2\pi x_3, \qquad (6) \end{aligned}$$

where $k_c = \pi/\sqrt{2}$. The horizontal components of velocity and vorticity fields are computed by the solenoidal characters of these two fields (i.e., $\nabla\cdot\mathbf{v} = \nabla\cdot\omega = 0$). We now project the hydrodynamical equations $(1-3)$ onto these twelve modes to get the following dynamical system

$$\tau\dot{\mathbf{X}} = \sigma(-\mathbf{X}+\mathbf{Y}) + \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} S_1 + \begin{pmatrix} G_2 \\ -G_1 \end{pmatrix} S_2 - \begin{pmatrix} G_2 \\ G_1 \end{pmatrix} V, \qquad (7)$$

$$\tau\dot{\mathbf{Y}} = -\mathbf{Y} + (r-Z)\mathbf{X} + \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} T_1 + \begin{pmatrix} G_2 \\ -G_1 \end{pmatrix} T_2, \qquad (8)$$

$$\tau\dot{\mathbf{G}} = -\sigma\mathbf{G} + \frac{2}{3}\begin{pmatrix} X_2 \\ -X_1 \end{pmatrix} S_2 + \begin{pmatrix} G_2 \\ G_1 \end{pmatrix} S_1, \qquad (9)$$

$$\tau\dot{\mathbf{S}} = -\frac{10}{3}\sigma\mathbf{S} + \frac{3}{5}\sigma\mathbf{T} - \begin{pmatrix} \frac{3}{10}(X_1X_2 + G_1G_2) \\ \frac{3}{40}(X_1G_2 - X_2G_1) \end{pmatrix}, \qquad (10)$$

$$\tau\dot{\mathbf{T}} = -\frac{10}{3}\mathbf{T} + r\mathbf{S} - \begin{pmatrix} \frac{1}{4}(X_1Y_2 + X_2Y_1) \\ \frac{3}{8}(Y_1G_2 - Y_2G_1) \end{pmatrix}, \qquad (11)$$

$$\tau\dot{V} = -\frac{2}{3}\sigma V + (X_1G_2 + X_2G_1), \qquad (12)$$

$$\tau\dot{Z} = -\frac{8}{3}Z + (X_1Y_1 + X_2Y_2), \qquad (13)$$

where the critical modes $\mathbf{X} \equiv (X_1,\ X_2)^T = \frac{\pi}{\sqrt{2}q_c^2}$ $(W_{101},\ W_{011})^T$, $\mathbf{Y} \equiv (Y_1,\ Y_2)^T = \frac{\pi {k_c}^2}{\sqrt{2}q_c^6}$ $(\Theta_{101},\ \Theta_{011})^T$, and $\mathbf{G} \equiv (G_1,\ G_2)^T = \frac{\pi}{\sqrt{2}q_c^3}$ $(\zeta_{101},\ \zeta_{011})^T$ are proportional to vertical velocity, temperature, and vertical vorticity respectively. The non-linear modes are redefined as $\mathbf{S} \equiv (S_1,\ S_2)^T = \frac{1}{4q_c}\ (\frac{\pi}{q_c}W_{112},\ W_{\bar{1}\bar{1}2})^T$, $V = \frac{\pi}{2q_c^3}\zeta_{110}$, $\mathbf{T} \equiv (T_1,\ T_2)^T = \frac{k_c^2}{4q_c^5}\ (\frac{\pi}{q_c}\Theta_{112},\ \Theta_{\bar{1}\bar{1}2})^T$, and $Z = -\frac{\pi k_c^2}{q_c^6}$ Θ_{002}. The constants of the model are $q_c^2 = \pi^2 + k_c^2$ and $\tau = q_c^{-2}$. Prandtl number σ and the reduced Rayleigh number $r = \frac{R}{R_c}\left(= \frac{Rk_c^2}{q_c^6}\right)$ are two bifurcation parameters of our model. the superscript T denotes the transpose of a matrix.

The model (7 - 13) describes various stationary as well as oscillating patterns on square lattice. The set of straight stripes (S) parallel to $x_{1(2)}$-axis is obtained by setting $X_{2(1)} = Y_{2(1)} = G_1 = G_2 = S_1 = S_2 = T_1 = T_2 = V = 0$ in the model. The stationary straight stripes given by $X_{1(2)} = Y_{1(2)} = \sqrt{8(r-1)/3}$, and $Z = r-1$ appear just above onset ($r = 1$) of convective instability. The stationary zig-zag (ZZ) patterns, which appear at secondary instability for ($\sigma < 1.57$), are obtained by taking $X_{2(1)} = Y_{2(1)} = G_{1(2)} = S_{1(2)} = T_{1(2)} = 0$ in the model. The standing asymmetric squares [18] is retrieved by setting $G_1 = G_2 = S_2 = T_2 = V = 0$ in the model. The asymmetric squares, which form mirror-symmetric patchwork quilt pattern, appear at the onset of secondary instability via forward Hopf bifurcation for $\sigma > 1.57$. The twelve-mode model describe interaction among these structures. We integrate numerically the full model to investigate dynamics of the resulting convective structures. We do it for a fixed σ by varying r in small steps. For each value of r, the integration is done starting with randomly chosen initial conditions for long enough to reach the final state. Prandtl number σ is then varied in small steps and whole procedure is repeated for each σ. The final states for various σ and r reported here are independent of the choice of initial conditions.

Figure 1 shows the stability boundaries of various patterns in parameter space ($\sigma - r$ plane) computed from the model. A transition from straight stripes (S) to zig-zag stripes (ZZ) occur as r is raised above its value at the lower stability boundary for $\sigma < 1.57$. The threshold value of r for such transition strongly depends on σ. The transition from straight stripe to standing patchwork quilt (PQ) via forward Hopf bifurcation occurs when r is raised above its value at the lower boundary for $\sigma > 1.57$. The patchwork quilt pattern shows

mirror and inversion symmetries but not four-fold symmetry. A shadowgraph of this pattern appears as standing asymmetric squares [18]. The stability boundary of this Hopf bifurcation shows weak dependence on σ.

All of the straight stripes (S), zig-zag stripes (ZZ) and standing patchwork quilt (PQ) are unstable in the region of parameter space marked as TPQ. We find all modes of the model active, if σ and r are chosen from this zone, and interacting with each other. We observe spatiotemporal patterns without mirror symmetry in this part of parameter space. Figure 2 shows the *twisted* patchwork quilt pattern slightly above the multicritical point. These patterns have lost the mirror symmetry. This happens due to competition of asymmetric squares, which are mirror symmetric patchwork quilt pattern, with zig-zag patterns. The generation of vertical vorticity breaks the mirror symmetry of patchwork quilt pattern (PQ) as σ and r are chosen from the zone marked TPQ in parameter space (see Fig.1). An increase in the intensity of vertical vorticity makes the pattern more twisted. The set of four figures clearly depicts the spatio-temporal behaviour of the texture of the *twisted* patchwork quilt patterns. The texture depends strongly on σ and weakly on r. Figure 3 shows competition of two sets, mutually perpendicular to each other, competing with each other. The picture for $\sigma = 0.6$ and $r = 9.0$ shows periodically varying textures arising due to competing instabilities for one period of oscillation. The model also shows chaotic patterns for $\sigma = 0.835$ and $r = 11.4$ (see fig. 4). This chaotic evolution of patterns occurs via quasi-periodic route.

In this article, we have presented a simple model of interacting instabilities. We have shown that the interaction between a stationary instability and an oscillatory instability may lead to many interesting patterns including *twisted* patchwork quilt on square lattice. The texture of the patterns due to competing instabilities may be modeled with an appropriate dynamical system. The model is also useful in capturing mechanism of emergence of various instabilities and resulting patterns.

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References

[1] P. Manneville, *Dissipative structures and weak turbulence* Acad. Press, San Diego (1990).

[2] M.C. Cross and P.C. and Hohenberg, *Rev. Mod. Phys.* **65**, 851 (1993).

[3] F.H. Busse, *Rep. Prog. Phys.* **41**, 1929 (1978).

[4] G. Ahlers, *Physica D* **51**, 421 (1991).

[5] S. Fauve, K. Kumar, C. Laroche, D. Beysens and Y. Garrabos, *Phys. Rev. Letts.* **68**, 3160 (1992).

[6] R.E. Ecke, Y. Hu, R. Mainieri and G. Ahlers, *Science* **269**, 1704 (1995).

[7] M. Assenheimer and V. Steinberg, *Phys. Rev. Lett.* **76**, 756 (1996).

[8] B.B. Plapp, D.A. Egolf, E. Bodenschatz and W. Pesh *Phys. Rev. Letts.* **81**, 5334 (1998).

[9] P.B. Umbanhowar, F. Melo and H.L. Swinney, *Nature* **382**, 793 (1996).

[10] A.T. Winfree, *Science* **266**, 1003 (1994).

[11] Q. Ouyang and J.F. Flesseles, *Nature* **379**, 143 (1996).

[12] M. Mitarai and H. Nakanishi, *Phys. Rev. Letts.* **85**, 1766 (2000).

[13] E.A. Brener and V.I. Mel'nikov, *Adv. Phys.* **40**, 53 (1991).

[14] J.V. Moloney and A.C. Newell, *Physica D* **44**, 1 (1990).

[15] E.N. Lorenz, *J. Atmos. Sci.* **20**, 130 (1963).

[16] M. Roberts, J.W. Swift and D.H. Wagner, *Contemp. Maths.* **56**, 283 (1985).

[17] J.S. Turner, *Annu. Rev. Fluid Mech.* **6**, 37 (1974).

[18] A. Das, U. Ghosal and K. Kumar, *Phys. Rev. E* **62**, R3051 (2000).

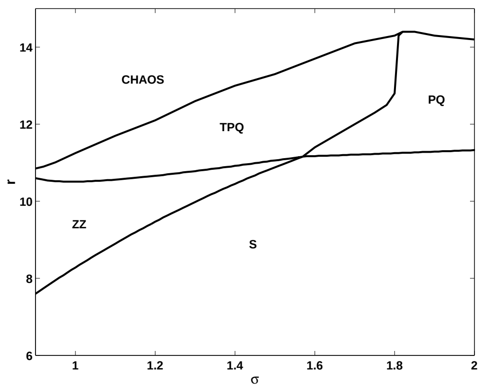

Figure 1: Stability boundaries of various convective structures in parameter space computed by the model. Stability zones of straight stripes (S), zig-zag stripes (ZZ), patchwork quilt (PQ), and *twisted* patchwork quilt (TPQ) meet at a multicritical point ($\sigma = 1.57 \pm 0.01,\ r = 11.2 \pm 0.05$). The model shows chaotic behaviour at much higher values of r.

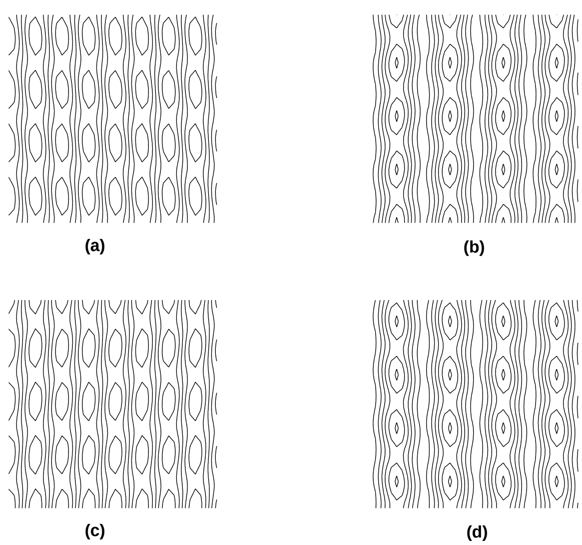

Figure 2: Texture of *twisted* patchwork quilt patterns. Stream lines for $\sigma = 1.57,\ r = 11.4, z = 0.25$ at (a) $t = 0$, (b) $t = T/4$, (c) $t = T/2$, and (d) $t = 3T/4$.

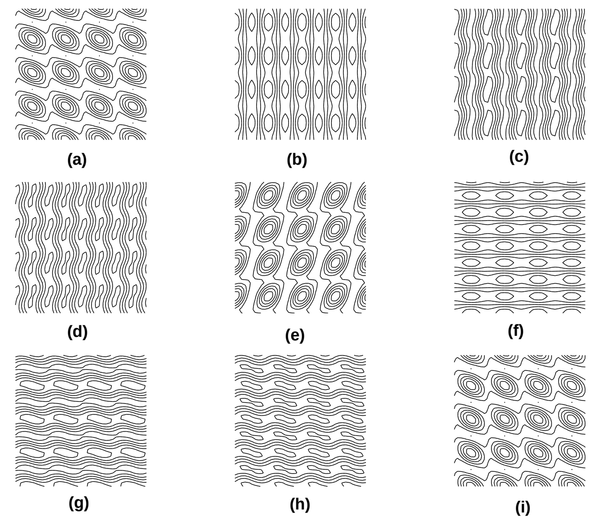

Figure 3: Temporal sequence of textures of structures due to competition of standing mirror-symmetric and *twisted patchwork quilt* patterns for $\sigma = 0.6,\ r = 9.0, z = 0.25$ at (a) $t = 0$, (b) $t = T/8$, (c) $t = T/4$, (d) $t = 3T/8$, (e) $t = T/2$, and (f) $t = 5T/8$, (g) $t = 3T/4$, (h) $t = 7T/8$, (i) $t = T$.

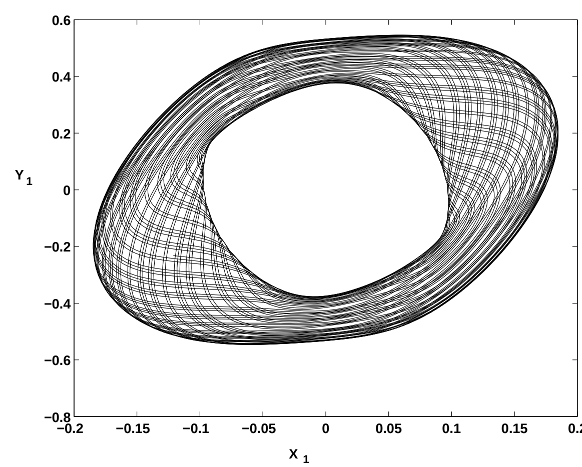

Figure 4: Chaotic patterns for $\sigma = 0.835,\ r = 11.4$. The chaos occurs via quasi-periodicity.